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Existence and blow up for a nonlinear hyperbolic equation with anisotropy ☆

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ABSTRACT

In this paper, we study a nonlinear wave equation with anisotropy and a source term. Under some appropriate assumptions on the parameters, and with certain initial data, we obtain several results on the existence of local solutions and the blow up of solutions in finite time.
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1. Introduction

We study the following wave equation with anisotropy

$$\begin{cases} u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) - \Delta u_t = g(u), & \text{in } [0,T)\times\Omega, \\ u = 0, & \text{on } [0,T)\times\partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & \text{in } \bar{\Omega}. \end{cases} \tag{1.1}$$

Here $p_i \geq 2$, $i = 1, \ldots, n$, $T > 0$, and Ω is a bounded open subset of $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega$. The function $g(u) = u|u|^{\sigma-2}$ is a polynomial source.

Problems related to the well-posedness of solutions to quasilinear wave equations

$$u_{tt} - \sum_{i=1}^{n}\{\sigma_i(u_{x_i})\}_{x_i} - \Delta u_t = f(x,t) \quad \text{in } (0,T)\times\Omega$$

have attracted a great deal of attention in the literature. Some important results can be found in [1–8]. Nonlinear hyperbolic equations of the type (1.1) have been investigated in the papers [9–16], just to cite a few. In these papers, by using Faedo–Galerkin approximation together with a combination of the compactness and the monotonicity methods, the authors obtained the existence of a global solution and derived decay results for the global solution. Problem (1.1) with $p_i = 2$ has been studied by many authors, notable among them is [17–20]. Gao and Ma [10] investigated the existence and asymptotic behavior of the solutions to problem (1.1) with $p_i = p > 2$. Sango [13] studied the existence of the solutions of the following

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initial boundary value problem

$$\begin{cases} u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) - \Delta u_t + g(x,u) = f(x,t), & \text{in } [0,T) \times \Omega, \\ u = 0, & \text{on } [0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.2}$$

where $p_i \geq 2$, $\Omega \subset R^n (n \geq 1)$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ the function $g(x,u)$ satisfies $|g(x,u)| \leq a|u|^{\sigma-1}$, $f(x,t)$ is a given function. The author proved the global existence of the solution to problem (1.2) when the parameters satisfy $1 < \sigma < \min\{p^*, p_1, \ldots, p_n\}$, and the function g satisfies $g(x,u)u \leq \delta G(x,u)$, where $G(x,u) = \int_0^u g(x,s)ds$, and $p^* = \frac{n\bar{p}}{n-\bar{p}}$, $\frac{1}{\bar{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}$. Closely related results may be found in the references cited in this paper and in [21,22].

It is well known that in (1.2), the damping term $-\Delta u_t$ plays a critical role in establishing the existence of solutions, and the attractive source term $g(u)$, i.e. $g(u)u \geq 0$, stabilizes the solution, while the forcing source $g(u)$, i.e. $g(u)u < 0$, destabilizes the solution and causes a finite time blow up.

Inspired by Sango [13], and Agre and Rammaha [23], the main aim of this paper is to give several results concerning the local existence and the finite time blow-up of weak solutions of (1.1). By using the arguments in [13,23], we prove that under suitable assumptions on σ, p_i, and with certain initial data, the solutions of (1.1) exist locally and blow up in finite time.

The rest of our paper is organized as follows. In Section 2, we give some notations and state our main results. Section 3 is devoted to proving our main results.

2. Notation and main results

Consider the anisotropic Sobolev space

$$W^1_{0,\tilde{p}}(\Omega) = \left\{ u \in W^{1,1}(\Omega) : u|_{\partial\Omega} = 0,\ \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega),\ i = 1, \ldots, n \right\},$$

with the norm

$$\|u\|_{W^1_{0,\tilde{p}}(\Omega)} = \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i},$$

where $\tilde{p} = (p_1, \ldots, p_n)$. We denote the dual of $W^1_{0,\tilde{p}}(\Omega)$ by $(W^1_{0,\tilde{p}}(\Omega))'$.

Now we set

$$\check{p} = \min\{p_1, \ldots, p_n\},$$
$$\hat{p} = \max\{p_1, \ldots, p_n\},$$
$$\check{p}^* = \frac{n\check{p}}{n-\check{p}} \quad \text{if } \check{p} < n.$$

For the parameter σ, we make the following assumptions

$$\begin{cases} 1 < \sigma < \check{p}^*, & n > \check{p} \\ 1 < \sigma < +\infty, & n \leq \check{p}. \end{cases} \tag{2.1}$$

We are now in a position to state our main results. Our first theorem establishes the existence of a local weak solution to (1.1).

Theorem 2.1. *Assume (2.1) holds, the parameters σ, p_i satisfy*

$$\frac{\hat{p}}{2} + 1 < \sigma < \frac{\check{p}^*}{2} + 1, \tag{2.2}$$

and further the initial data u_0, u_1 satisfy the conditions

$$u_0 \in W^1_{0,\tilde{p}}(\Omega), \qquad u_1 \in L^2(\Omega).$$

Then there exists a weak solution $u(x,t)$ to (1.1), such that

$$u \in L^\infty(0,T; W^1_{0,\tilde{p}}(\Omega)); \qquad u_t \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; L^2(\Omega)),$$

for some $T > 0$.

Remark 1. When we take $p_i = 2$, (2.2) becomes $2 < \sigma < \frac{2n-2}{n-2}$, this condition coincides with some results in [17]; when we take $p_i = p > 2$, (2.2) becomes $\frac{p}{2} + 1 < \sigma < \frac{np}{2(n-p)} + 1$, which partially covers some results in [10].

Our next theorem addresses the issue of a strong source which may lead to a finite-time blow up of solutions.

Theorem 2.2. *Suppose that σ satisfies*

$$\hat{p} \le \sigma < \frac{\check{p}^*}{2} + 1. \tag{2.3}$$

Let u be a solution of (1.1) with initial data satisfying

$$E(0) \le 0, \qquad (\sigma - 2)\int_\Omega u_0 u_1 dx > 2\|\nabla u_0\|_2^2. \tag{2.4}$$

Then the solution of (1.1) blows up in finite time T^*, i.e.,

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \longrightarrow +\infty, \quad \text{as } t \to T^*, \tag{2.5}$$

where

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \sum_{i=1}^n \frac{1}{p_i}\left\|\frac{\partial u_0}{\partial x_i}\right\|_{p_i}^{p_i} - \frac{1}{\sigma}\|u_0\|_\sigma^\sigma.$$

3. Proof of main results

In this section, we prove the existence of the solution to problem (1.1). First we give a lemma.

Lemma 3.1. *Let $a_1, \ldots, a_n > 0$, $p_i \ge 2$, for $i = 1, \ldots, n$. Then the following inequality holds:*

$$\sum_{i=1}^n a_i \le C\left(\sum_{i=1}^n a_i^{p_i}\right)^{1/\check{p}} + C\left(\sum_{i=1}^n a_i^{p_i}\right)^{1/\hat{p}}, \tag{3.1}$$

where C is dependent on $p_i (i = 1, 2, \ldots, n)$.

Proof. (i) $a_i \ge 1$. We have

$$a_i \le a_i^{p_i/\check{p}} = (a_i^{p_i})^{1/\check{p}}. \tag{3.2}$$

(ii) $a_i < 1$. We have

$$a_i \le a_i^{p_i/\hat{p}} = (a_i^{p_i})^{1/\hat{p}}. \tag{3.3}$$

From (3.2) and (3.3), we easily get

$$a_i \le (a_i^{p_i})^{1/\check{p}} + (a_i^{p_i})^{1/\hat{p}}. \tag{3.4}$$

Summing (3.4) over i, we have

$$\begin{aligned}\sum_{i=1}^n a_i &\le \sum_{i=1}^n (a_i^{p_i})^{1/\check{p}} + \sum_{i=1}^n (a_i^{p_i})^{1/\hat{p}} \\ &\le C\left(\sum_{i=1}^n a_i^{p_i}\right)^{1/\check{p}} + C\left(\sum_{i=1}^n a_i^{p_i}\right)^{1/\hat{p}},\end{aligned}$$

where C is dependent on $p_i (i = 1, 2, \ldots, n)$. The proof is completed. □

Proof of Theorem 2.1. Let r be such that $\frac{r-1}{n} > \frac{1}{2} - \frac{1}{\hat{p}}$. This choice of $\hat{p}$ implies that $H_0^r(\Omega) \subset W_0^{1,\hat{p}}(\Omega)$, while $W_0^{1,\hat{p}}(\Omega) \subset W_{0,\vec{p}}^1(\Omega)$. So the embedding $H_0^r(\Omega) \subset W_{0,\vec{p}}^1(\Omega)$ holds.

Using the method similar to [13], let ω_j $(j \in N)$ be eigenfunctions of the spectral problem

$$(\omega_j, v)_{H_0^r(\Omega)} = \lambda_j(\omega_j, v)_{L^2(\Omega)}, \quad \text{for all } v \in H_0^r(\Omega).$$

Then the family $\{\omega_1, \ldots, \omega_m, \ldots\}$ yields a basis for $H_0^r(\Omega)$. For each $m \in N$, let V_m be the span of $\{\omega_1, \ldots, \omega_m\}$. We look for an approximate solution to our problem in the form

$$u_m(t) = \sum_{j=1}^m k_{jm}(t)\omega_j,$$

where the functions k_{jm} are derived from the system of nonlinear ordinary differential equations in t

$$\int_\Omega \left\{u_{mtt} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u_m}{\partial x_i}\right|^{p_i-2} \frac{\partial u_m}{\partial x_i}\right) - \Delta u_{mt}\right\} \omega_j dx = \int_\Omega u_m|u_m|^{\sigma-2}\omega_j dx \quad (j = 1, \ldots, m), \tag{3.5}$$

with the initial conditions

$$u_m(0) = u_{0m}; \qquad u_{mt}(0) = u_{1m},$$

where u_{0m} and u_{1m} are chosen in V_m such that

$$u_{0m} \longrightarrow u_0 \quad \text{strongly in } W^1_{0,\tilde{p}}(\Omega),$$

and

$$u_{1m} \longrightarrow u_1 \quad \text{strongly in } L^2(\Omega).$$

By using the standard Picard iteration method, we know that the system admits a solution on some interval $[0, T_m)$, for every $m \geq 1$ and some $T_m > 0$.

A priori estimate I:

Multiplying (3.5) by $k'_{jm}(t)$ and summing the resulting equations over j, we get after integration by parts

$$\frac{d}{dt}\left[\frac{1}{2}\|u_{mt}\|_2^2 + \sum_{i=1}^n \frac{1}{p_i}\left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}^{p_i}\right] + \|\nabla u_m(t)\|_2^2 = \int_\Omega u_m|u_m|^{\sigma-2}u_{mt}dx. \tag{3.6}$$

Taking integration over $[0, t]$,

$$\begin{aligned}\frac{1}{2}\|u_{mt}(t)\|_2^2 &+ \sum_{i=1}^n \frac{1}{p_i}\left\|\frac{\partial u_m(t)}{\partial x_i}\right\|_{p_i}^{p_i} + \int_0^t \|\nabla u_m(s)\|_2^2 ds \\ &= \frac{1}{2}\|u_{1m}\|_2^2 + \sum_{i=1}^n \frac{1}{p_i}\left\|\frac{\partial u_{0m}}{\partial x_i}\right\|_{p_i}^{p_i} + \int_0^t\int_\Omega u_m(s)|u_m(s)|^{\sigma-2}u_{mt}(s)dxds.\end{aligned} \tag{3.7}$$

For the last term of (3.7), we do the following estimates:

$$\begin{aligned}\left|\int_\Omega u_m|u_m|^{\sigma-2}u_{mt}dx\right| &\leq \int_\Omega |u_m|^{\sigma-1}|u_{mt}|dx \\ &\leq \frac{1}{2}\|u_m\|_{2(\sigma-1)}^{2(\sigma-1)} + \frac{1}{2}\|u_{mt}\|_2^2.\end{aligned} \tag{3.8}$$

By using (2.2), and the Sobolev embedding inequality, we have

$$\|u_m\|_{2(\sigma-1)} \leq C\|u_m\|_{W_0^{1,\check{p}}(\Omega)} \leq C\|u_m\|_{W^1_{0,\tilde{p}}(\Omega)}. \tag{3.9}$$

From (3.9), we can get

$$\|u_m\|_{2(\sigma-1)}^{2(\sigma-1)} \leq C_1\left(\sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}\right)^{2(\sigma-1)}, \tag{3.10}$$

using (3.1), we have

$$\left(\sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}\right) \leq \left(\sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{1/\check{p}} + \left(\sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{1/\hat{p}}, \tag{3.11}$$

and the following estimate holds

$$\left(\sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}\right)^{2(\sigma-1)} \leq C\left[\left(\sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\check{p}} + \left(\sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\hat{p}}\right]. \tag{3.12}$$

Combining (3.10) and (3.12), we have

$$\|u_m\|_{2(\sigma-1)}^{2(\sigma-1)} \leq C\left(\sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\check{p}} + C\left(\sum_{i=1}^n \left\|\frac{\partial u_m}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\hat{p}}. \tag{3.13}$$

Substitute (3.8) and (3.13) into (3.7), we can deduce

$$\begin{aligned}
&\frac{1}{2}\|u_{mt}(t)\|_2^2+\sum_{i=1}^{n}\frac{1}{p_i}\left\|\frac{\partial u_m(t)}{\partial x_i}\right\|_{p_i}^{p_i}+\int_0^t\|\nabla u_m(s)\|_2^2ds\\
&\quad\le\frac{1}{2}\|u_{1m}\|_2^2+\sum_{i=1}^{n}\frac{1}{p_i}\left\|\frac{\partial u_{0m}}{\partial x_i}\right\|_{p_i}^{p_i}+\frac{1}{2}\int_0^t\|u_m(s)\|_{2(\sigma-1)}^{2(\sigma-1)}ds+\frac{1}{2}\int_0^t\|u_{mt}(s)\|_2^2ds\\
&\quad\le\frac{1}{2}\|u_{1m}\|_2^2+\sum_{i=1}^{n}\frac{1}{p_i}\left\|\frac{\partial u_{0m}}{\partial x_i}\right\|_{p_i}^{p_i}+\frac{1}{2}\int_0^t\|u_{mt}(s)\|_2^2ds\\
&\qquad+\int_0^t\left\{C\left(\sum_{i=1}^{n}\left\|\frac{\partial u_m(s)}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\check{p}}+C\left(\sum_{i=1}^{n}\left\|\frac{\partial u_m(s)}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\hat{p}}\right\}ds.
\end{aligned}\tag{3.14}$$

Define $y_m(t)=\|u_{mt}(t)\|_2^2+\sum_{i=1}^{n}\|\frac{\partial u_m(s)}{\partial x_i}\|_{p_i}^{p_i}+1$. Then we consider the following two cases:

Case (i) $\sum_{i=1}^{n}\|\frac{\partial u_m(s)}{\partial x_i}\|_{p_i}^{p_i}\ge 1$.

Then we have

$$\left(\sum_{i=1}^{n}\left\|\frac{\partial u_m(s)}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\check{p}}+\left(\sum_{i=1}^{n}\left\|\frac{\partial u_m(s)}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\hat{p}}\le 2\left(\sum_{i=1}^{n}\left\|\frac{\partial u_m(s)}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\check{p}},\tag{3.15}$$

by using (3.15), (3.14) becomes

$$y_m(t)+C_1\int_0^t\|\nabla u_m(s)\|_2^2ds\le C_0+C\int_0^t[y_m(s)]^{2(\sigma-1)/\check{p}}ds.\tag{3.16}$$

In particular, $y_m(t)$ satisfies

$$y_m(t)\le C_0+C\int_0^t[y_m(s)]^a ds,\tag{3.17}$$

where $a=2(\sigma-1)/\check{p}$, and by (2.2), $a>1$. Hence, by using a standard comparison theorem, (3.17) yields that

$$y_m(t)\le z(t),$$

where $z(t)=[C_0^{1-a}-C(a-1)t]^{\frac{1}{1-a}}$ is the solution of the Volterra integral equation

$$z(t)=C_0+C\int_0^t z(t)^a ds.\tag{3.18}$$

Although $z(t)$ blows up in finite time, nonetheless, there exists a time $0<T<T_m$ such that

$$y_m(t)\le z(t)\le C_1,\quad \text{for all } t\in[0,T],$$

where C_1 is independent of m. Hence, for all $m\ge 1$, one has

$$y_m(t)\le C_1,\quad \text{for all } t\in[0,T].\tag{3.19}$$

Combining (3.16) and (3.19), we easily have

$$\int_0^t\|\nabla u_m(s)\|_2^2ds\le C,\quad \text{for all } t\in[0,T].\tag{3.20}$$

Case (ii) $\sum_{i=1}^{n}\|\frac{\partial u_m(s)}{\partial x_i}\|_{p_i}^{p_i}<1$,

then

$$\left(\sum_{i=1}^{n}\left\|\frac{\partial u_m(s)}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\check{p}}+\left(\sum_{i=1}^{n}\left\|\frac{\partial u_m(s)}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\hat{p}}\le 2\left(\sum_{i=1}^{n}\left\|\frac{\partial u_m(s)}{\partial x_i}\right\|_{p_i}^{p_i}\right)^{2(\sigma-1)/\hat{p}}.$$

Similar to the discussion in Case (i), we have

$$y_m(t)\le C_1,\quad \text{for all } t\in[0,T],$$

and

$$\int_0^t \|\nabla u_m(s)\|_2^2 ds \le C, \quad \text{for all } t \in [0,T].$$

Using a similar priori estimate II to [13], the proof of Theorem 2.1 is completed. $\square$

In order to prove Theorem 2.2, we quote a lemma as follows.

Lemma 3.2 *([24]). Assume that $P(t) \in C^2$, $P(t) \ge 0$, satisfies the inequality*

$$P(t)P''(t) - (1+\delta)P'^2(t) \ge 0,$$

for certain real number $\delta > 0$, and $P(0) > 0$, $P'(0) > 0$. Then there exists a real number $\tilde{T}$ such that $0 < \tilde{T} \le P(0)/\delta P'(0)$ and

$$P(t) \to \infty \quad \text{as } t \to \tilde{T}^-.$$

Proof of Theorem 2.2. Define the general energy $E(t)$ of (1.1) as

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \sum_{i=1}^n \frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{p_i} - \frac{1}{\sigma}\|u\|_\sigma^\sigma. \tag{3.21}$$

By a simple computation, we can get

$$E'(t) = -\|\nabla u_t\|_2^2 \le 0. \tag{3.22}$$

Assume by contradiction that the solution u of (1.1) is global. Then we consider a function $G : [0, T_0] \longrightarrow R^+$ defined by

$$G(t) = \|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds + (T_0 - t)\|\nabla u_0\|_2^2, \quad t \le T_0,$$

where the parameter T_0 is a positive constant to be determined later.

Notice that $G(t) > 0$ for all $t \in [0, T_0]$; hence, since G is continuous, there exists a constant $\rho > 0$ such that

$$G(t) \ge \rho \quad \text{for all } t \in [0, T_0]. \tag{3.23}$$

Furthermore,

$$\begin{aligned} G'(t) &= 2\int_\Omega u u_t dx + \|\nabla u(t)\|_2^2 - \|\nabla u_0\|_2^2 \\ &= 2\int_\Omega u u_t dx + 2\int_0^t \int_\Omega \nabla u(s)\cdot\nabla u_t(s) dx ds, \end{aligned}$$

and, consequently, using Eq. (1.1), we obtain

$$G''(t) = 2\|u_t\|_2^2 - 2\sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{p_i} + 2\|u\|_\sigma^\sigma.$$

Therefore, using the Hölder inequality we get

$$\begin{aligned} G'(t)^2 &= 4\left(\int_\Omega u u_t dx + \int_0^t \int_\Omega \nabla u(s)\cdot\nabla u_t(s) dx ds\right)^2 \\ &= 4\left[\left(\int_\Omega u u_t dx\right)^2 + 2\left(\int_\Omega u u_t dx\right)\int_0^t\int_\Omega \nabla u(s)\cdot\nabla u_t(s)dxds + \left(\int_0^t\int_\Omega \nabla u(s)\cdot\nabla u_t(s)dxds\right)^2\right] \\ &\le 4\left[\|u_t\|_2^2\|u\|_2^2 + 2\|u\|_2^2\int_0^t \|\nabla u_t(s)\|_2^2 ds + \|u_t\|_2^2\int_0^t \|\nabla u(s)\|_2^2 ds\right. \\ &\quad \left. + \left(\int_0^t \|\nabla u(s)\|_2^2 ds\right)\left(\int_0^t \|\nabla u_t(s)\|_2^2 ds\right)\right] \\ &= 4\left(\|u\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds\right)\left(\|u_t\|_2^2 + \int_0^t \|\nabla u_t(s)\|_2^2 ds\right) \\ &\le 4G(t)\left(\|u_t\|_2^2 + \int_0^t \|\nabla u_t(s)\|_2^2 ds\right). \end{aligned} \tag{3.24}$$

Using (3.21) and (3.22), we can have

$$\begin{aligned} G''(t)G(t) - \frac{\sigma+2}{4}G'(t)^2 &\geq \left(G''(t) - (\sigma+2)\left(\|u_t\|_2^2 + \int_0^t \|\nabla u_t(s)\|_2^2 ds\right)\right) G(t) \\ &= \left[-\sigma\|u_t\|_2^2 - 2\sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{p_i} + 2\|u\|_\sigma^\sigma - (\sigma+2)\int_0^t \|\nabla u_t(s)\|_2^2 ds\right] G(t) \\ &= \left[-2\sigma E(0) + (\sigma-2)\int_0^t \|\nabla u_t(s)\|_2^2 ds + \sum_{i=1}^n \frac{2(\sigma-p_i)}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{p_i}\right] G(t). \end{aligned} \tag{3.25}$$

By using (2.3), (3.23), and $E(0) \leq 0$, it follows that

$$G''(t)G(t) - \frac{\sigma+2}{4}G'(t)^2 \geq -2\rho\sigma E(0), \quad \text{for } t \in [0, T_0].$$

Moreover, $G(0) > 0$. From (2.4), we can check that $G'(0) > 0$, and we can choose T_0 satisfying

$$\|u_0\|_2^2 < T_0\left(\frac{\sigma-2}{2}\int_\Omega u_0u_1dx - \|\nabla u_0\|_2^2\right).$$

According to Lemma 3.2, there exists a real number T^* such that $T^* < G(0)/\delta G'(0)$ and $T^* < T_0$, and we have

$$\lim_{t\to T^{*-}} G(t) = \infty,$$

i.e.,

$$\lim_{t\to T^{*-}}\left(\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds\right) = \infty.$$

This completes the proof of Theorem 2.2. □

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